EIGENVALUES OF SOME COMPOSITE GRAPHS

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Abstract

K_n is a complete graph with n vertices. $K_n^{(m)}$ is a graph containing m copies of K_n with each vertex of a K_n is only adjacent to a vertex of each of the other K_n.

We will show that the adjacency matrix of $K_n^{(m)}$ has

(i) $(n-1)(m-1)$ eigenvalues of -2

(ii) $m-1$ eigenvalues of $n-2$

(iii) $n-1$ eigenvalues of $m-2$

(iv) an eigenvalue of $n+m-2$.

Composite Graph $K_n^{(m)}$ and Its Adjacency Matrix

K_n is a complete graph with n vertices. $K_n^{(m)}$ is a graph containing m copies of K_n with each vertex of a K_n is only adjacent to a vertex of each of the other K_n. (See figure 1 for $K_3^{(4)}$)

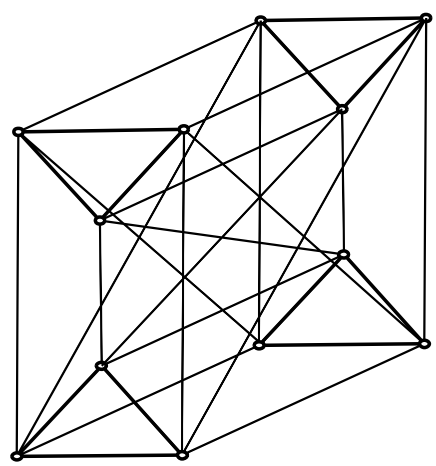

Figure 1. $K_3^{(4)}$

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Basic definitions and notations on graphs and their eigenvalues can be found in [1, 2, 3] and others.

The adjacency matrix of $K_n^{(m)}$ is like as follow:

$$\begin{bmatrix} K_n & I_n & I_n & \cdots & I_n \\ I_n & K_n & I_n & \cdots & I_n \\ I_n & I_n & K_n & \cdots & I_n \\ I_n & I_n & I_n & \ddots & I_n \\ I_n & I_n & I_n & I_n & K_n \end{bmatrix}$$

For example, adjacency matrix of $K_3^{(4)}$ is

$$\left[\begin{array}{ccc:ccc:ccc:ccc} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Since adjacency matrix $\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \ddots & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ of K_n is like as $\begin{bmatrix} K_1 & I_1 & I_1 & \cdots & I_1 \\ I_1 & K_1 & I_1 & \cdots & I_1 \\ I_1 & I_1 & K_1 & \cdots & I_1 \\ I_1 & I_1 & I_1 & \ddots & I_1 \\ I_1 & I_1 & I_1 & \cdots & K_1 \end{bmatrix}$,

$K_n^{(m)}$ can be seen as a generalization of complete graphs.

Eigenvalues and Eigenvectors of Adjacency Matrix of $K_n^{(m)}$

The adjacency matrix of $K_n^{(m)}$ has

(i) $(n-1)(m-1)$ eigenvalues of -2

(ii) $m-1$ eigenvalues of $n-2$

(iii) $n-1$ eigenvalues of $m-2$

(iv) an eigenvalue of $n+m-2$.

By using each of the eigenvectors shown in figure 2, one can check that there are $(n-1)(m-1)$ eigenvalues of -2.

$$
\begin{bmatrix}1\\-1\\0\\0\\\vdots\\0\\0\\-1\\1\\0\\0\\\vdots\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\\vdots\\0\\0\\0\\0\\\vdots\\0\\0\end{bmatrix}
\begin{bmatrix}1\\-1\\0\\0\\\vdots\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\-1\\1\\0\\0\\\vdots\\0\\0\\\vdots\\0\\0\\0\\0\\\vdots\\0\\0\end{bmatrix}\cdots
\begin{bmatrix}1\\-1\\0\\0\\\vdots\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\\vdots\\-1\\1\\0\\0\\\vdots\\0\\0\end{bmatrix},
\begin{bmatrix}1\\0\\-1\\0\\\vdots\\0\\0\\-1\\0\\1\\0\\\vdots\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\\vdots\\0\\0\\0\\0\\\vdots\\0\\0\end{bmatrix}
\begin{bmatrix}1\\0\\-1\\0\\\vdots\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\-1\\0\\1\\0\\\vdots\\0\\0\\\vdots\\0\\0\\0\\0\\\vdots\\0\\0\end{bmatrix}\cdots
\begin{bmatrix}1\\0\\-1\\0\\\vdots\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\\vdots\\-1\\0\\1\\0\\\vdots\\0\\0\end{bmatrix},\ldots,
\begin{bmatrix}1\\0\\0\\0\\\vdots\\0\\-1\\-1\\0\\0\\0\\\vdots\\0\\1\\0\\0\\0\\0\\\vdots\\0\\0\\\vdots\\0\\0\\0\\0\\\vdots\\0\\0\end{bmatrix}
\begin{bmatrix}1\\0\\0\\0\\\vdots\\0\\-1\\0\\0\\0\\0\\\vdots\\0\\0\\-1\\0\\0\\0\\\vdots\\0\\1\\\vdots\\0\\0\\0\\0\\\vdots\\0\\0\end{bmatrix}\cdots
\begin{bmatrix}1\\0\\0\\0\\\vdots\\0\\-1\\0\\0\\0\\0\\\vdots\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\\vdots\\-1\\0\\0\\0\\\vdots\\0\\1\end{bmatrix}
$$

Figure 2. $(n-1)(m-1)$ eigenvectors of eigenvalue -2

$$\left[\begin{array}{c}-1\\-1\\\vdots\\-1\\\hdashline 1\\1\\\vdots\\1\\\hdashline 0\\0\\\vdots\\0\\\hdashline\vdots\\\hdashline 0\\0\\\vdots\\0\end{array}\right],\left[\begin{array}{c}-1\\-1\\\vdots\\-1\\\hdashline 0\\0\\\vdots\\0\\\hdashline 1\\1\\\vdots\\1\\\hdashline\vdots\\\hdashline 0\\0\\\vdots\\0\end{array}\right],\cdots,\left[\begin{array}{c}-1\\-1\\\vdots\\-1\\\hdashline 0\\0\\\vdots\\0\\\hdashline 0\\0\\\vdots\\0\\\hdashline\vdots\\\hdashline 1\\1\\\vdots\\1\end{array}\right],\qquad \left[\begin{array}{c}-1\\1\\0\\0\\\vdots\\0\\0\\\hdashline -1\\1\\0\\0\\\vdots\\0\\\hdashline\vdots\\\hdashline -1\\1\\0\\0\\\vdots\\0\\0\end{array}\right],\left[\begin{array}{c}-1\\0\\1\\0\\\vdots\\0\\0\\\hdashline -1\\0\\1\\0\\\vdots\\0\\\hdashline\vdots\\\hdashline -1\\0\\1\\0\\\vdots\\0\\0\end{array}\right],\cdots,\left[\begin{array}{c}-1\\0\\0\\0\\\vdots\\0\\1\\\hdashline -1\\0\\0\\0\\\vdots\\1\\\hdashline\vdots\\\hdashline -1\\0\\0\\0\\\vdots\\0\\1\end{array}\right],\qquad \left[\begin{array}{c}1\\1\\\vdots\\1\\1\\1\\\vdots\\1\\1\\1\\\vdots\\1\\\vdots\\1\\1\\\vdots\\1\end{array}\right]$$

(a) (b) (c)

Figure 3. (a) $m-1$ eigenvectors of eigenvalue $n-2$;
(b) $n-1$ eigenvectors of eigenvalue $m-2$; (c) an eigenvector of eigenvalue $n+m-2$.

According to eigenvectors shown in figure 3, there are $m-1$ eigenvalues of $n-2$, $n-1$ eigenvalues of $m-2$ and an eigenvalue of $n+m-2$.

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